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A Complete Set of Translation Invariants Based on the Cyclic Correlation Property of the Generalized Circular Transforms

Volker Lohweg, Dietmar Müller*

Linnenstrasse 35, D-33699 Bielefeld
E-mail: v.lohweg@owl-online.de
**Professur für Schaltungs- und Systementwurf*
Technical University of Chemnitz, D-09107 Chemnitz, Germany

Abstract

Nonlinear spatial transforms are established in feature extraction and pattern recognition. This paper describes a method for calculating the cyclic crosscorrelation with the help of generalized circular transforms. Based on the crosscorrelation property, a nonlinear translation invariant autocorrelation and a complete set of simplified translation invariants, called $\tilde{\boldsymbol{G}}$*-spectrum, can be deduced. Moreover, the generalized circular transforms can now be interpreted as signal dependent transforms. In addition we will demonstrate how the* $\tilde{\boldsymbol{G}}$*-spectrum can significantly extend the pattern separability properties for transforms with low computeral complexity.*

1. List of Acronyms

$\mathbb{C}T$	A class of nonlinear translation invariant transforms.
FPGA	Field Programmable Gate Array.
GT	Generalized Transforms. Members are the Walsh transform and the Fourier Transform.
MGT	Modified Generalized Transforms
GCT	Generalized Circular Transform. A class of nonlinear translation invariant transforms.
GCTn	A member of the GCT.
MWHT	Modified Walsh Hadamard Transform.
RMWHT	Rationalized Modified Walsh Hadamard Transform.
RT	Rapid Transform. A member of $\mathbb{C}T$.
SWT	Square Wave Transform.
WHT	Walsh Hadamard Transform.

2. Introduction

Nonlinear spatial transforms are established in signal processing. They have proved to be helpful tools in feature extraction and pattern recognition. Well known members of the above mentioned nonlinear transforms are the $\mathbb{C}T$-transforms (e.g. Rapid transform), the power spectrum of the Fourier transform, invariant integration and others; s.f. [4,5,7,10,11,12]. Our approach is based on nonlinear translation invariant transforms. In [7,8] classes of one- and two-dimensional transforms are described which can be easily calculated. The class of the transform is based on a general concept. This concept uses the so called *generalized characteristic* and *generalized circular* matrices. Therefore they are called *generalized circular transforms*. All transforms have in common that they use an amplitude spectrum $\boldsymbol{G}$ with $\mathrm{ld}(N)+1$ coefficients (1D-case). The coefficients are ordered in period groups, similar to the power spectrum of the Walsh Hadamard transform [2][3]. In opposite to the power spectrum of the WHT, we have developed an interesting property which holds for all the generalized circular transforms. An *absolute value spectrum* $\boldsymbol{G}$ is defined, which operates with sums of ordered period groups. Absolute values of spectral coefficients are summed up periodwise. This property is not valid for the WHT and the generalized transforms (GT, MGT) [3].

An interesting fact is that the modified Walsh Hadamard transform (MWHT) [3] and the *Square Wave transform* (SWT) recently proposed by Pender and Covey [6], have also this above mentioned property. This paper describes a method to calculate the cyclic crosscorrelation with the help of generalized circular transforms (GCT)[8]. Based on the crosscorrelation property, a translation invariant autocorrelation and a complete set of simplified translation invariants, called *extended absolute value spectrum* $\tilde{\boldsymbol{G}}$-spectrum, can be deduced. Implementation in radix-2-structure is possible with a computational complexity of $\boldsymbol{O}(N)$ up to $\boldsymbol{O}(N\mathrm{ld}(N))$, where N is the length of the input data vector in the 1D-case. Hence, a hardware implementation on FPGA is easy to achieve [8,9]. This paper focuses on the general concept of the cyclic correlation property in the 1D-case. The 2D-case can be deduced accordingly. Introducing the correlation property of the GCT, we can give a new interpretation for the transform matrix coefficients. Moreover, the generalized circular transforms can now be interpreted as signal dependent transforms. In addition we compared different pattern separability properties of the $\boldsymbol{G}$-, and $\tilde{\boldsymbol{G}}$-spectrum as well as the autocorrelation for binary pattern.

The separability properties can be drastically extended. Moreover, the separability properties of the Fourier power spectrum can be determined with the transform coefficients {+1,-1}.

3. Circular Transforms

In this section we sum up the major properties of the generalized circular transforms [8]. Let $x_i \in \mathbb{R}$ and $\boldsymbol{x}^T = \{x_0, x_1, ..., x_{N-1}\}$ be an input vector (one-dimensional) and $\boldsymbol{X}^T = \{X_0, X_1, ..., X_{N-1}\}$ its transformed output vector. With $\boldsymbol{A}_N$ and $\boldsymbol{B}_N$ we denote the spatial circular transform and its inverse ($\boldsymbol{A}$ and $\boldsymbol{B}$ are quadratic (N xN)-transform matrices). The transform matrices define a biorthogonal basis set.

$$\boldsymbol{X} = \boldsymbol{A}_N \cdot \boldsymbol{x} \text{ and } \boldsymbol{x} = \tfrac{1}{N} \cdot \boldsymbol{B}_N^T \cdot \boldsymbol{X}\,. \tag{1}$$

Given a (2x2)-Hadamard matrix $\boldsymbol{K} = \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}$ [1]. The transform matrices can be expressed and evaluated recursively as:

$$\boldsymbol{A}_N = \boldsymbol{diag}({}^f\boldsymbol{T}_{\frac{N}{2}}, \boldsymbol{A}_{\frac{N}{2}}) \cdot \left[\boldsymbol{K} \otimes \boldsymbol{I}_{\frac{N}{2}}\right],$$

$$\boldsymbol{B}_N = \boldsymbol{diag}({}^r\boldsymbol{T}_{\frac{N}{2}}, \boldsymbol{B}_{\frac{N}{2}}) \cdot \left[\boldsymbol{K} \otimes \boldsymbol{I}_{\frac{N}{2}}\right]. \tag{2}$$

The generalized characteristic matrices ${}^f\boldsymbol{T}$ and ${}^r\boldsymbol{T}$ are defined for the dimension $(\frac{N}{2} x \frac{N}{2})$. The characteristic matrices are calculated recursively. For $\frac{N}{2}$ the characteristic matrix is defined as follows:

$${}^f\boldsymbol{T}_{\frac{N}{2}} := \begin{bmatrix} -\beta_{\frac{N}{2}-1} & \cdots & -\beta_0 \\ \vdots & \ddots & \vdots \\ \beta_{\frac{N}{2}-2} & \cdots & -\beta_{\frac{N}{2}-1} \end{bmatrix}. \tag{3}$$

The coefficient matrix $\boldsymbol{A}$ can now be defined in a sparse matrix form:

$$\boldsymbol{A}_N := \begin{bmatrix} {}^f\boldsymbol{T}_{\frac{N}{2}} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & 1 \end{bmatrix} \cdot ...$$

$$... \cdot \left[\prod_{i=1}^{\mathrm{ld}(N)-1} \boldsymbol{diag}(\boldsymbol{I}_{N-2^i}, \boldsymbol{K} \otimes \boldsymbol{I}_{2^{i-1}})\right] \cdot \left[\boldsymbol{K} \otimes \boldsymbol{I}_{\frac{N}{2}}\right]. \tag{4}$$

The last two matrices represent the rationalized MWHT [7]. Eq.(4) shows that it is possible to characterize the generalized circular transforms, with all the above mentioned properties, with only one *characteristic coefficient vector*:

$$\boldsymbol{c}_\beta = \left\{\beta_{\frac{N}{2}-1}, \beta_{\frac{N}{2}-2}, ..., \beta_0, \beta_{\frac{3 \cdot N}{4}-1}, ..., \beta_{\frac{N}{2}}, ..., \beta_{N-1}\right\}^T. \tag{5}$$

Using different transform kernels $\boldsymbol{T}$, it is possible to generate various properties for the transforms. The spectral coefficients of all $\boldsymbol{A}$ and $\boldsymbol{B}$ transforms are grouped in the same way: Begining with the first $N/2$ spectral coefficients with the period N, $N/4$ coefficients with the period $N/2$ following. The last two coefficient vectors are the vectors with the shortest possible period 2 and the vector with the period 0. The last vector denotes the average value of the input vector.

3.1 Shift Matrix and Absolute Value Spectrum *G*

With $\boldsymbol{G}$ we denote the translation invariant absolute value spectrum. It is defined by the above mentioned period groups, nearly the same way as the power spectrum of the WHT. We have used the well known concept of calculating the shift matrix ${}^s\boldsymbol{S}_N := \frac{1}{N} \cdot \boldsymbol{A}_N \cdot {}^s\boldsymbol{I}_N \cdot \boldsymbol{B}_N^T$, $-(N-1) \le s \le (N-1)$ [2] of a transform. It can be shown for all circular transforms that the addition of all absolute values of the spectral vector leads to a translation invariant spectrum $\boldsymbol{G}$. [7,8].

4. The Cyclic Correlation Property

The general idea is based on the calculation of the cyclic spectral cross- and auto- correlation of the circular transforms. Let $x_i, y_i \in \mathbb{R}$ and $\boldsymbol{x}^T = \{x_0, x_1, ..., x_{N-1}\}$, resp. $\boldsymbol{y}^T = \{y_0, y_1, ..., y_{N-1}\}$ be the input vectors. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two real N-periodic signals, then the 1D-discrete cyclic crosscorrelation in the spatial domain is defined as follows [3]:

$$\boldsymbol{r}_{xy} := \frac{1}{N} \cdot \begin{bmatrix} y_0 & y_1 & \cdots & y_{N-1} \\ y_{N-1} & y_0 & & y_{N-2} \\ \vdots & & \ddots & \vdots \\ y_1 & y_2 & \cdots & y_0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$$= \frac{1}{N} \cdot \hat{\boldsymbol{y}} \cdot \boldsymbol{x},\ \hat{\boldsymbol{y}} = \begin{bmatrix} \boldsymbol{C}_{\frac{N}{2}} & \boldsymbol{D}_{\frac{N}{2}} \\ \boldsymbol{D}_{\frac{N}{2}} & \boldsymbol{C}_{\frac{N}{2}} \end{bmatrix} \quad . \qquad (6)$$

In the spectral domain the cyclic crosscorrelation can be written as $\boldsymbol{R}_{xy} = \boldsymbol{A}_N \cdot \boldsymbol{r}_{xy} = \frac{1}{N^2} \cdot \boldsymbol{A}_N \cdot \hat{\boldsymbol{y}} \cdot \boldsymbol{B}_N^T \cdot \boldsymbol{X}$. The matrix $\hat{\boldsymbol{y}}$ can be divided into submatrices, called $\boldsymbol{C}$ and $\boldsymbol{D}$. Applying eq.(2) to $\boldsymbol{R}_{xy}$ results in

$$\boldsymbol{R}_{xy} = \frac{1}{N^2} \cdot \underbrace{\begin{bmatrix} \boldsymbol{M}_\Delta & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_\Sigma \end{bmatrix}}_{\Delta_N} \cdot \boldsymbol{X} \ .$$

$$\boldsymbol{M}_\Delta := 2 \cdot {}^f\boldsymbol{T}_{\frac{N}{2}} \cdot (\boldsymbol{C}_{\frac{N}{2}} - \boldsymbol{D}_{\frac{N}{2}}) \cdot {}^r\boldsymbol{T}_{\frac{N}{2}}^T \ ,$$

$$\boldsymbol{M}_\Sigma := 2 \cdot {}^f\boldsymbol{T}_{\frac{N}{2}} \cdot (\boldsymbol{C}_{\frac{N}{2}} + \boldsymbol{D}_{\frac{N}{2}}) \cdot {}^r\boldsymbol{T}_{\frac{N}{2}}^T \ . \qquad (7)$$

With $\Delta_{\frac{N}{2}} = (\boldsymbol{C}_{\frac{N}{2}} - \boldsymbol{D}_{\frac{N}{2}})$ and $\Sigma_{\frac{N}{2}} = (\boldsymbol{C}_{\frac{N}{2}} + \boldsymbol{D}_{\frac{N}{2}})$ we denote a difference matrix and a sum matrix which combines the y_i values in a particular way. Analyzing eq.(7) and the above mentioned matrices leads to the following statements:

I. $2 \cdot {}^f\boldsymbol{T}_{\frac{N}{2}} \cdot (\boldsymbol{C}_{\frac{N}{2}} - \boldsymbol{D}_{\frac{N}{2}}) \cdot {}^r\boldsymbol{T}_{\frac{N}{2}}^T \equiv \Delta_{\frac{N}{2}}$ and $\Sigma_{\frac{N}{2}} \equiv \boldsymbol{R}_{xy_{\frac{N}{2}}}$.

II. The matrix Δ_N has block-diagonal structure and can be created in a recursion process.

III. The first rows of each block Δ_i of the matrix Δ_N can be determined by the calculation of the rationalizied MWHT (RMWHT): $\boldsymbol{Y} = \text{RMWHT} \cdot \boldsymbol{y}$.

IV. The generalized characteristic matrix $\boldsymbol{T}$ is signal dependent and can be determined via the matrix Δ_N.

The cyclic crosscorrelation in the spectral domain can now be written as:

$$\boldsymbol{R}_{xy} = \begin{bmatrix} \Delta_{\frac{N}{2}} & & & \boldsymbol{0} \\ & \Delta_{\frac{N}{4}} & & \\ & & \ddots & \\ \boldsymbol{0} & & & 1 \end{bmatrix} \cdot \boldsymbol{X} \ . \qquad (8)$$

To get a further insight, we compared the result of eq.(8) with the generalized characteristic matrix $\boldsymbol{T}$ and the characteristic coefficient vector $\boldsymbol{c}_\beta$. It follows, that the elements of the block-diagonal $\boldsymbol{T}$ matrix can be determined by the Δ matrix. Each block is calculated independently.Hence, $\{\beta_{k-1}, \beta_{k-2}, ..., \beta_0\} = -\{Y_0, Y_1, ..., Y_{k-1}\}$, the coefficients of the $\boldsymbol{T}$ matrix can be calculated via the RMWHT. Until now, no constraints were defined for the spectral coefficients $\boldsymbol{X}$. It is evident (s. f. eq.(4)) to calculate the coefficients $\boldsymbol{X}$ with the RMWHT, because the computeral complexity is extremely low. Of course any other transform is applicable. The result in the spectral domain can be written in a compact expression. Here is the result for the first block $\{X_0, ..., X_{(N/2)-1}\}$ of the Δ matrix.. The results for the other blocks can be determined accordingly.

$$R_{xy_0}^{(N/2)} = \frac{1}{N^2} \sum_{i=0}^{\frac{N}{2}-1} X_i \cdot Y_i,\ p = 0,$$

$$R_{xy_p}^{(N/2)} = \frac{1}{N^2} \left[\sum_{i=0}^{\frac{N}{2}-1-p} X_i \cdot Y_{i+p} - \sum_{i=0}^{p-1} X_{i+\frac{N}{2}-p} \cdot Y_i \right],$$

$$p = 1, 2, ..., \frac{N}{2} - 1. \qquad (9)$$

Remark. The *cyclic convolution* is obtained by applying the shift matrix ${}^s\boldsymbol{S}_i$ for s=1 to the spectral coefficients for each period group $\{\beta_0, \beta_1, ...\}^T = {}^1\boldsymbol{S}_i \cdot \{Y_0, Y_1, ...\}^T$.

4.1 The Translation Invariant Autocorrelation

A complete set of translation invariants can be obtained by using the discrete *cyclic autocorrelation* [3] $\boldsymbol{R}_{xx} = \boldsymbol{R}_{xy}\big|_{Y_q \Leftarrow X_q}, q = 0, 1, ..., N-1$ of a data sequence. Again, we used the well known concept of calculating the shift matrix ${}^s\boldsymbol{S}_N$ of a transform. If the data sequence in the spatial domain is permuted cyclicly, then the submatrices Δ_i contain all shift combinations of the spectral vector $\boldsymbol{X}$. It should be noted, that the spectrum $\boldsymbol{R}_{xx}$ contains N spectral coefficients, but only $(\frac{N}{2} + 1)$

coefficients are needed to represent the cyclic translation invariant autocorrelation. The other coefficients are redundant. The autocorrelation sequence possesses odd symmetry of the coefficients in each period group. In particular, each coefficient group is odd around its midpoint. The midpoint is always zero. Therefore it is only necessary to compute the first half of the coefficients in each group.

4.2 The Extended Absolute Value Spectrum $\tilde{\boldsymbol{G}}$

Based on the cyclic autocorrelation we introduce a translation invariant spectrum with reduced computeral complexity. The spectrum $\tilde{\boldsymbol{G}}$ can favourably be utilized in pattern recognition. The idea is to substitute one of the spectral coefficient vectors in each term of the autocorrelation function by its binarized value +1 or -1. This is achieved by introducing the signum function to the elements of the Δ matrix. Hence:

$$\tilde{\boldsymbol{G}} = \boldsymbol{R}_{xy}\Big|_{Y_q \Leftarrow \operatorname{sgn}(X_q)} \,, q = 0, 1, ..., N-1. \qquad (10)$$

From eq.(10) it follows that $\left\{\tilde{G}_0, \tilde{G}_{\frac{N}{4}}, ..., \tilde{G}_{N-1}\right\}$ represents the known $\boldsymbol{G}$-spectrum.

5. Results

In this section we present some experimental results using the transforms for pattern separability tests. We used binary test patterns as input vectors. Binary numbers can be interpreted as patterns under cyclic permutation. If a left (or right) shift is used on a particular number, a new number in the particular class is generated (e.g. N = 16; number of separable patterns = 4116). We compared our results with the results given by the well known Fourier transform power spectrum and the Rapid transform spectrum. In addition, we used four circular transforms $\tilde{\boldsymbol{G}}$ (GCT1, GCT2, GCT3 [8], SWT [6]).

The circular transforms are defined (example for N = 16) as follows:

1. GCT1: $\boldsymbol{c}_{\beta 1} = \{2^7, 2^6, \ldots, 2^0, 2^3, 2^2, \ldots, 2^0, 0, -1, -1, 1\}^T$

This circular transform has a computational complexity of Nld(N). All computations can be calculated with integers.

2. GCT2: $\boldsymbol{c}_{\beta 2} = -k \cdot \cos(\frac{\pi \cdot (i + \frac{1}{2})}{N})$ with $i = 1, 2, \ldots, N-1$.

A radix-2-structure is not possible. The factor k is chosen in a way that the last spectral coefficient represents the average value of the input vector.

3. GCT3: $\boldsymbol{c}_{\beta 3} = \{r_0, r_1, ..., r_{N-1}\}^T$.

The GCT3 coefficients are defined as a Gaussian noise signal with variance $\sigma = 1$ and average = 0. Calculations in radix-2-structure are not possible.

4. SWT: $\boldsymbol{c}_{\beta 4} = \{1, 1, \ldots, 1, 1, 1, \ldots, 1, 1, -1, -1, 1\}^T$.

This circular transform has a computational complexity of Nld(N). All computations can be calculated with binary integers (here: +1,-1).

It is evident that the autocorrelation coefficients of all the above mentioned transforms separate the same amount (1876) of binary patterns as the Fourier power spectrum. The proposed circular transforms $\boldsymbol{G}$-, and $\tilde{\boldsymbol{G}}$ -spectrum are superior in comparison to the Fourier power spectrum and the Rapid transform for N > 4. In general, the separability properties of the $\tilde{\boldsymbol{G}}$-spectrum of the GCT1, GCT2 and GCT3 are some what better than those of the $\boldsymbol{G}$-spectrum. The separability properties of the SWT is increased significantly, where as the computeral complexity of the SWT is low compared to that of GCT1, GCT2 and GCT3. On the other hand, the $\boldsymbol{G}$-spectrum can be determined with ld(N)+1 coefficients, where as the $\tilde{\boldsymbol{G}}$-spectrum needs N coefficients (1D case). Table 1 and 2 shows the results of the separability tests.

Table 1. $\boldsymbol{G}$-spectrum: Amount of separable binary patterns

N	2^N	*Number of Sep. Patterns*	*Rapid Transform*	*Fourier Power Spectrum*
4	16	6	6	6
8	256	36	21	31
16	65536	4116	225	1876

N	2^N	*GCT1*	*GCT2*	*GCT3*	*SWT* [6]
4	16	6	6	6	6
8	256	31	31	33	29
16	65536	3245	3496	3527	668

Table 2. $\tilde{G}$-spectrum: Amount of separable binary patterns

N	2^N	*Number of Sep. Patterns*	*Rapid Transform.*	*Fourier Power Spectrum*
4	16	6	6	6
8	256	36	21	31
16	65536	4116	225	1876

N	2^N	*GCT1*	*GCT2*	*GCT3*	***SWT*** [6]
4	16	6	6	6	6
8	256	33	33	33	33
16	65536	3527	3527	3527	3527

6. Conclusion

Based on the cyclic correlation property of the signal dependend generalized circular transforms, we have presented an extended translation invariant spectrum $\tilde{G}$, which can be calculated with simple operations, such as addition, subtraction and some absolute value calculations. Furthermore, since no transcedental functions appear in the GCT correlation method, there is no round-off error in these algorithms. Unlike the Fourier transform, no complex calculations are needed. The proposed concept leads to a significant increased separability property for the SWT. The computeral complexity is low compared to other transforms.

7. References

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